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NEW ELEMENT FORMULATION FOR FREE VIBRATION ANALYSIS OF TIMOSHENKO BEAM ON PASTERNAK ELASTIC FOUNDATION

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ABSTRACT

In this paper, the free vibration of the beam on elastic foundation is studied using finite element method. To this end, a two- node Timoshenko element is used for beam modeling. Each node in this element has a rotational and translational degree of freedom, which encompasses all four degrees of freedom. The displacement and rotational fields of this beam is selected from the third and the second order, respectively. Moreover, the shear strain of the element is assumed as a constant value. Interpolation functions for displacement field and beam rotation are explicitly calculated by employing total beam energy and it's stationary with respect to shear strain. Also, two-parameter elastic foundation model is used. In this method, the soil is modeled as a layer of Winkler springs with a shear layer on it. Next, by utilizing the interpolation functions, the stiffness matrices of beam and foundation, as well as their mass matrices are introduced; hence, the free vibration analysis on the elastic foundation is carried out. Finally, after conducting several tests, the high efficiency and accuracy of the proposed element is demonstrated.

Keywords: Finite element; free vibration; Timoshenko beam; Pasternak; elastic foundation.

1. INTRODUCTION

In the analysis of such structures as buildings foundations, highways, and railways, there is a need for modeling the beams on elastic foundation. In order to simplify the modeling of soil-foundation interaction, soil environment is assumed homogeneous and isotropic with linear elastic behavior. Beam modeling on elastic foundation have been extensively presented in previous studies; in order to model the foundation of structures, models such as Winkler,

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Pasternak, Vlasov, and Flonenko-Borodich are currently being used. As the simplest model for modeling foundation, Winkler is presented in 1867, in which the relationship between pressure and vertical displacement of the foundation is modeled as linear springs [1]. In contrast, in Pasternak's two-parameter model, soil is modeled as a layer of Winkler springs as well as a shear layer. Therefore, the latter provides a more accurate method compared to simpler models like the Winkler foundation.

So far, several studies have been carried out for the analysis of free vibration. Zhou studied the free vibration of beams on Winkler elastic foundation [2]. Eisenberger investigated the free vibration response of the beam on Winkler elastic foundation with variable stiffness [3]. Also, Eisenberger and Clastorink examined the buckling and vibration of beams on these foundations [4]. Further studies on the Winkler foundation can be found in Refs. [5] and [6]. Auersch carried out a study about infinite beams on half-space compared with finite and infinite beams on a Winkler support [7]. These models are typically used in the analysis of railways. A study of infinite beam models, giving importance on asymptotic behavior at high frequencies, was conducted by Ruge and Birk [8]. Oz and Pakdemirli examined the resonances of shallow beams resting on elastic foundations [9].

Up to now, few studies have been carried out on the Pasternak foundation compared to that of Winkler. De Rosa and Maurizi [10] calculated the exact free vibration frequencies of an Euler beam on elastic foundation. They modeled the soil environment as a two-parameter elastic medium. Franciosi and Masi [11] performed the free vibration analysis of Euler-Bernoulli beam on Pasternak foundation using finite element analysis. Using Green's functions, Wang et al. [12] offered an exact solution of the free vibration of Timoshenko beam on Pasternak foundation. Chen et al. [13] proposed a mixed method based on differential quadrature (DQ) formulation, for bending and free vibration of arbitrarily thick beams resting on a Pasternak elastic foundation. Chen [14] developed a new differential quadrature element method (DQEM) for free vibration analysis of prismatic beam on an elastic foundation. Calio and Greco [15] carried out the free vibration and stability of axially-loaded Timoshenko beams on Pasternak foundation through dynamic stiffness matrix method. Other researchers, too, have studied the analysis of elastic foundations with Winkler-Pasternak models, Refs. [16, 17].

In this paper, the free vibration of Timoshenko beam on Pasternak foundation is studied by finite element method. To this end, an element beam with two nodes was first modeled so that each node has a transitional and rotational degree of freedom. The displacement and rotational fields of this element were selected from the third and the second order respectively. The shear strain of the element was assumed constant. Having considered the total strain energy of the beam element as stationary with respect to the shear strain, the shape functions were calculated for the displacement field and the beam rotation. Using these interpolation functions, stiffness and mass matrices of the beam as well as the stiffness matrix of the foundation were calculated. At the end, compared with other methods, the accuracy and efficiency of the proposed element was confirmed via numerical methods.

2. TWO-NODE BEAM ELEMENT FORMULATION

In the finite element formulation, the displacement and rotation functions are related to the nodal degrees of freedom by shape functions. The shape functions of the Timoshenko beam were calculated based on Fig. 1. In order to compute the shape function of the beam in Fig. 1, a cubic displacement polynomial and a quadratic rotational field were selected. Moreover, it was assumed that the shear strain has a constant value of γ_0. Based on these, the following equations can be assumed:

$$w = \frac{w_i}{2}(1-s) + \frac{w_j}{2}(1+s) + \beta_0 l\left(1-s^2\right) + \beta_1 ls\left(1-s^2\right) \tag{1}$$

$$\theta = \frac{\theta_i}{2}(1-s) + \frac{\theta_j}{2}(1+s) + \alpha_0\left(1-s^2\right) \tag{2}$$

$$\gamma = \gamma_0 \tag{3}$$

$$s = \frac{2x}{l} - 1 \tag{4}$$

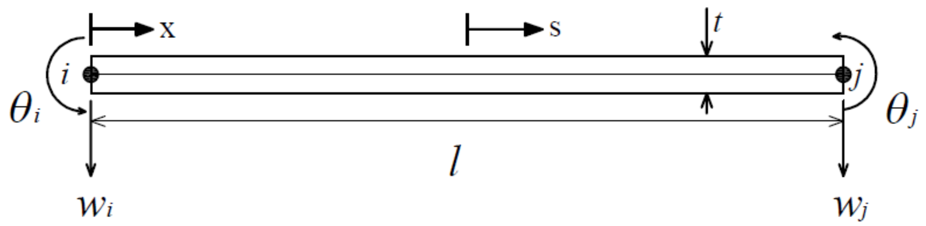


Figure 1.Timoshenko beam element

In these relations, $\beta_1, \beta_0, \alpha_0$ and γ_0 are unknown parameters. In order to determine their values, the equation of shear strain for Timoshenko beam was first established. By considering the shear strain value equal to γ_0, the subsequent equalities will be available:

$$\gamma = \frac{dw}{dx} - \theta = \frac{2}{l}\cdot\frac{dw}{ds} - \theta \tag{5}$$

$$\gamma_0 = \frac{2}{l}\left(-\frac{w_i}{2} + \frac{w_j}{2} - 2\beta_0 ls + \beta_1 l - 3\beta_1 ls^2\right) - \theta_i\left(\frac{1-s}{2}\right) - \theta_j\left(\frac{1+s}{2}\right) - \alpha_0\left(1-s^2\right) \tag{6}$$

In the relation (6), coefficients of the terms s and s^2 are equivalent to zero. Therefore, in the following lines, α_0, β_1 are determined in terms of the unknown parameter γ_0:

$$\Gamma = \frac{2}{l}\left(w_j - w_i\right) - \left(\theta_i + \theta_j\right) \tag{7}$$

$$\beta_0 = \frac{1}{8}\left(\theta_i - \theta_j\right) \tag{8}$$

$$\alpha_0 = -\frac{3}{2}\left(\gamma_0 - \frac{1}{2}\Gamma\right) \tag{9}$$

$$\beta_1 = \frac{1}{6}\alpha_0 \tag{10}$$

At this stage, there is only one unknown constant γ_0 which can be determined from the condition of minimum strain energy. It should be added that the structural strain energy is the sum of bending and shear strain energy. Bending strain energy is calculated in the following way:

$$U_b = \frac{EI}{2}\int_0^l \kappa^2 dx = \frac{EIl}{4}\int_{-1}^1 \kappa^2 ds \tag{11}$$

In this equation, EI and κ represents the stiffness and curvature of beam, respectively. The curvature κ is determined as below:

$$\kappa = -\frac{2}{l}\cdot\frac{d\theta}{ds} = \kappa_0 - 6\frac{s\gamma_0}{l} \tag{12}$$

$$\kappa_0 = \frac{1}{l}\left(\theta_i - \theta_j + 3s\Gamma\right) \tag{13}$$

Substituting these equations into (11) leads to the below bending strain energy:

$$U_b = U_0 + 6\left(-\frac{D\Gamma\gamma_0}{l} + \frac{D\gamma_0^2}{l}\right) \tag{14}$$

$$U_0 = \frac{Dl}{4}\int_{-1}^1 \kappa_0^2\, ds \tag{15}$$

Besides, the next equation determines the energy of shear strain:

$$U_s = \frac{GA}{2f_s}\int_0^l \gamma^2\, dx = \frac{GAl}{4f_s}\int_{-1}^1 \gamma_0^2\, ds = \frac{GAl}{2f_s}\gamma_0^2 \tag{16}$$

In the equations (16), f_s is shear correction factor, which is equal to 6/5 for beam with rectangular cross section. By adding the bending and the shear strain energy together, total strain energy was found as follows:

$$U = U_b + U_s = U_0 - \frac{6EI\Gamma}{l}\gamma_0 + \frac{6EI}{l}\gamma_0^2 + \frac{GAl}{2f_s}\gamma_0^2 \tag{17}$$

Implementing $\partial U/\partial\gamma_0 = 0$ will give the following results:

$$\gamma_0 = \frac{6EI\Gamma}{\frac{GAl^2}{f_s} + 12EI} = \delta\Gamma \tag{18}$$

$$\delta = \frac{6\lambda}{l^2 + 12\lambda} \ , \ \lambda = \frac{f_s EI}{GA} \tag{19}$$

Substitution of $\beta_1, \beta_0, \alpha_0$ and γ_0 into relations (1) and (2) results in the following interpolation functions for Timoshenko beam:

$$\begin{Bmatrix} w \\ \theta \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \\ N_5 & N_6 & N_7 & N_8 \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} \tag{20}$$

$$\begin{aligned}
N_1 &= \frac{1}{4}\left[2 + s^3(1-2\delta) + s(-3+2\delta)\right] \\
N_2 &= \frac{l}{4}\left[0.5(1-s^2) + (s^3 - s)(0.5-\delta)\right] \\
N_3 &= \frac{1}{4}\left[2 - s^3(1-2\delta) - s(-3+2\delta)\right] \\
N_4 &= \frac{l}{4}\left[-0.5(1-s^2) + (s^3 - s)(0.5-\delta)\right] \\
N_5 &= \frac{1}{4l}\left[6(1-s^2)(-1+2\delta)\right] \\
N_6 &= \frac{1}{4}\left[-1 + s(-2+3s) + 6(1-s^2)\delta\right] \\
N_7 &= \frac{1}{4l}\left[-6(1-s^2)(-1+2\delta)\right] \\
N_8 &= \frac{1}{4}\left[-1 + s(2+3s) + 6(1-s^2)\delta\right]
\end{aligned} \tag{21}$$

Therefore, by using the interpolation $[N]$, the strain field $\{\varepsilon\}$ and the strain matrix $[B]$ were obtained as follows:

$$\{\varepsilon\}=\begin{Bmatrix}\dfrac{dw}{dx}\\ -\dfrac{dw}{dx}+\theta\end{Bmatrix}=[B]\{D\}_E \tag{22}$$

$$[B]=\begin{bmatrix}0 & \dfrac{d}{dx}\\ \dfrac{d}{dx} & -1\end{bmatrix}[N] \tag{23}$$

In the equation (22), $\{D\}_E$ is the nodal displacement.

3. EQUATIONS OF MOTION

Fig. 2 shows the Timoshenko beam on Pasternak foundation. In this model, the normal stress $\sigma(x,t)$ and vertical displacement $w(x,t)$ at an arbitrary point of the lower boundary of the beam provide the following relation [13]:

$$\sigma(x,t)=K_w w(x,t)-K_p\frac{\partial^2 w(x,t)}{\partial x^2} \tag{24}$$

Where K_w and K_p are the Winkler and shearing layer elastic foundation moduli, respectively. The shearing soil parameter K_p can be estimated as [11]:

$$K_p=\frac{E_s}{4(1+\nu_s)}\frac{B}{\beta} \tag{25}$$

Where B is the width of the beam and β is a parameter which characterizes the distribution of displacements along the vertical direction in the elastic foundation. Also, E_s and ν_s are Young's modulus and Poisson's ratio of soil, respectively.

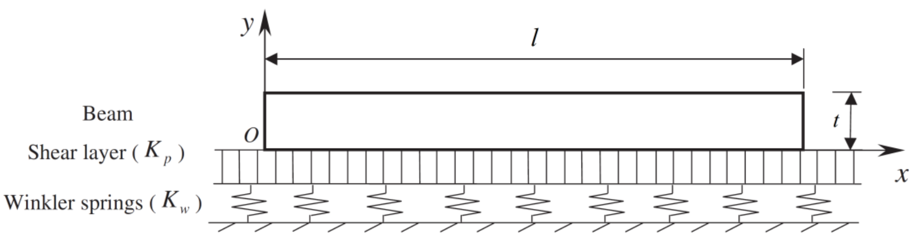


Figure 2.Geometry of a beam on Pasternak foundation

For a uniform beam resting on two-parameter foundation, the total potential energy can be obtained as follows:

$$\Pi = \Delta U + T \tag{26}$$

$$\Delta U = \frac{EI}{2}\int_0^l \kappa^2 dx + \frac{GA}{2f_s}\int_0^l \gamma^2 dx + \frac{K_p}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx + \frac{K_w}{2}\int_0^l w^2 dx \tag{27}$$

$$T = \frac{1}{2}\int_{-l/2}^{l/2} \rho A \dot{w}^2 dx + \frac{1}{2}\int_{-l/2}^{l/2} \rho I \dot{\theta}^2 dx \tag{28}$$

In the equation (28), $\dot{w}$ and $\dot{\theta}$ are respectively the derivations of translation and rotation fields with respect to time. By substituting the relations (20) to (22) in the above equations, the internal and kinematic energy can be evaluated as:

$$\begin{aligned}\Delta U = {} & \frac{\{D\}_E^T}{2}\int_0^l [B]^T [D_m][B] dx \{D\}_E + \frac{K_p \{D\}_E^T}{2}\int_0^l [A]^T [A] dx \{D\}_E \\ & + \frac{K_w \{D\}_E^T}{2}\int_0^l [N]^T [k][N] dx \{D\}_E\end{aligned} \tag{29}$$

$$T = \{\dot{D}\}_E^T \left(\frac{l}{2}\int_1^1 \rho A [N_w]^T [N_w] ds + \frac{l}{2}\int_{-l/2}^{l/2} \rho I [N_\theta]^T [N_\theta]^T ds\right)\{\dot{D}\}_E \tag{30}$$

The elasticity matrix $[D_m]$ for the Timoshenko beam has the following shape:

$$[D_m] = \begin{bmatrix} EI & 0 \\ 0 & \dfrac{GA}{f_s} \end{bmatrix} \tag{31}$$

Introducing the total potential energy (Π) into Hamilton's principle leads to the matrix equation governing the free vibrations of the Timoshenko beam on the Pasternak elastic foundation as follows:

$$[M]\{\ddot{D}\}_E + [K]\{D\}_E = 0 \tag{32}$$

Where, $[K]$ and $[M]$ are the stiffness and mass matrices, respectively. The stiffness matrix is given by:

$$\begin{aligned}[K] = [K_B] + [K_{F1}] + [K_{F2}] = {} & \int_0^l [B]^T [D_m][B] dx \\ & + K_p \int_0^l [A]^T [A] dx + K_w \int_0^l [N_w]^T [N_w] dx\end{aligned} \tag{33}$$

The beam stiffness matrix $[K_B]$ was obtained as:

$$[K_B]=\frac{EI}{l^3+12l\lambda}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2+12\lambda & -6l & 2l^2-12\lambda \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2-12\lambda & -6l & 4l^2+12\lambda \end{bmatrix} \tag{34}$$

Also, the stiffness matrix of shear layer of foundation $[K_{F1}]$ was calculated as below:

$$[K_{F1}]=\frac{K_p l}{30\left(l^3+12l\lambda\right)^2}\begin{bmatrix} k_{f\,11} & k_{f\,12} & -k_{f\,11} & k_{f\,12} \\ -k_{f\,11} & k_{f\,13} & -k_{f\,12} & k_{f\,14} \\ k_{f\,23} & -k_{f\,12} & k_{f\,11} & -k_{f\,12} \\ k_{f\,12} & k_{f\,14} & -k_{f\,12} & k_{f\,13} \end{bmatrix} \tag{35}$$

The entries of this matrix can be defined as follows:

$$\begin{aligned} k_{f\,11} &= 36l^4+720\lambda l^2+4320\lambda^2 \\ k_{f\,12} &= 3l^5 \\ k_{f\,13} &= 4l^6+60\lambda l^4+360\lambda^2 l^2 \\ k_{f\,14} &= -l^6-60\lambda l^4-360\lambda^2 l^2 \end{aligned} \tag{36}$$

Additionally, the stiffness matrix of Winkler layer of foundation $[K_{F2}]$ was obtained as follows:

$$[K_{F2}]=\frac{K_w l^3}{420\left(l^3+12l\lambda\right)^2}\begin{bmatrix} k_{f\,21} & k_{f\,22} & k_{f\,23} & k_{f\,24} \\ k_{f\,22} & k_{f\,25} & -k_{f\,24} & k_{f\,26} \\ k_{f\,23} & -k_{f\,24} & k_{f\,21} & -k_{f\,22} \\ k_{f\,24} & k_{f\,26} & -k_{f\,22} & k_{f\,25} \end{bmatrix} \tag{36}$$

In the following, the entries of this matrix are introduced:

$$\begin{aligned}
k_{f\,21} &= 156 l^4 + 3528\lambda l^2 + 20160\lambda^2 \\
k_{f\,22} &= 22 l^5 + 462\lambda l^3 + 2520\lambda^2 l \\
k_{f\,23} &= 54 l^4 + 1512\lambda l^2 + 10080\lambda^2 \\
k_{f\,24} &= -13 l^5 - 378\lambda l^3 - 2520\lambda^2 l \\
k_{f\,25} &= 4 l^6 + 84\lambda l^4 + 504\lambda^2 l^2 \\
k_{f\,26} &= -3 l^6 - 84\lambda l^4 - 504\lambda^2 l^2
\end{aligned} \tag{37}$$

The mass matrix of the beam element $[M]$ in relation (32) includes the two following parts, one related to translations and the other to rotations:

$$[M] = [M_1] + [M_2] = \frac{l}{2}\int_{1}^{1} \rho A [N_w]^T [N_w] ds + \frac{l}{2}\int_{-l/2}^{l/2} \rho I [N_\theta]^T [N_\theta]^T ds \tag{38}$$

In this equation, ρ is the mass density of the material of the beam and I is the second moment of area of cross section. The translation mass matrix $[M_1]$ was obtained as follows:

$$[M_1] = \frac{\rho A l^5}{210\left(12\lambda + l^2\right)^2} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{15} & -m_{14} & m_{16} \\ m_{13} & -m_{14} & m_{11} & -m_{12} \\ m_{14} & m_{16} & -m_{12} & m_{15} \end{bmatrix} \tag{39}$$

The entries of this matrix can be defined as:

$$\begin{aligned}
m_{11} &= \frac{6}{l^4}\left(1680\lambda^2 + 294 l^2\lambda + 13 l^4\right) \\
m_{12} &= \frac{1}{l^3}\left(1260\lambda^2 + 231 l^2\lambda + 11 l^4\right) \\
m_{13} &= \frac{9}{l^4}\left(560\lambda^2 + 84 l^2\lambda + 3 l^4\right) \\
m_{14} &= -\frac{1}{2 l^3}\left(2520\lambda^2 + 378 l^2\lambda + 13 l^4\right) \\
m_{15} &= \frac{2}{l^2}\left(126\lambda^2 + 21 l^2\lambda + 13 l^4\right) \\
m_{16} &= -\frac{3}{2 l^2}\left(168\lambda^2 + 28 l^2\lambda + 13 l^4\right)
\end{aligned} \tag{40}$$

In addition, the rotation mass matrix $[M_2]$ was obtained as follows:

$$[M_2] = \frac{\rho I l^2}{30(12\lambda + l^2)^2} \begin{bmatrix} m_{21} & m_{22} & -m_{21} & m_{22} \\ m_{22} & m_{23} & -m_{22} & m_{24} \\ -m_{21} & -m_{22} & m_{21} & -m_{12} \\ m_{22} & m_{24} & -m_{12} & m_{23} \end{bmatrix} \tag{41}$$

Where

$$\begin{aligned} m_{21} &= 36l \\ m_{22} &= -3(60\lambda - l^2) \\ m_{23} &= \frac{4}{l}(360\lambda^2 + 15l^2\lambda + 3l^4) \\ m_{24} &= \frac{1}{l}(720\lambda^2 - 60l^2\lambda - l^4) \end{aligned} \tag{42}$$

By assembling the mass and stiffness matrix of the elements, the total mass and stiffness matrix can be obtained, respectively. For free vibration analysis, the assembled nodal displacement vector $\{D\}$ is assumed to be harmonic in time with circular frequency ω as follows:

$$\{D\} = \sin(\omega t)\{W\} \tag{43}$$

Where $\{W\}$ is the vector of nodal displacement amplitudes of vibration. By substituting the relation (43) into (32), the following eigenvalue is obtained:

$$([K - \omega^2 [M]])\{W\} = \{0\} \tag{44}$$

For the non-trial solution of equation (44), it is essential that the determinant of matrix $([K - \omega^2 [M]])$ be zero at the correct natural frequencies. Therefore, the values of natural frequency ω were obtained.

4. NUMERICAL TESTS

In this section, the efficiency and accuracy of the proposed element for free vibration analysis of beams on elastic foundation was investigated. To achieve this goal, the dimensionless frequency for the proposed element was compared with the results of previous studies. The dimensionless frequency Ω is defined as:

$$\Omega = \sqrt{\omega l^2 \sqrt{\frac{\rho A}{EI}}} \tag{45}$$

Also, the dimensionless elastic and shear modulus of the foundations are defined as follows:

$$\bar{K}_w = \frac{K_w l^4}{EI} \quad , \quad \bar{K}_p = \frac{K_p l^2}{\pi^2 EI} \tag{46}$$

To reveal the accuracy and efficiency of the proposed element, three tests were conducted in the present paper; as the first test, the effect of the number of degrees of freedom on the accuracy of free vibration of a simply supported thin beam (t/l =0.001) on the elastic foundation was examined. The shear and elastic modulus of the soil were $\bar{K}_p$ = 2.5 and $\bar{K}_w$ = 10^6 respectively. The results of this analysis along with those of the previous studies are represented in Table (1).

Table 1: The first three natural frequency parameters (Ω) for simply supported beam

DOFs	**Exact (Rosa & Maurizi [10])**	**Franciosi & Masi [11]**	**Proposed element**
	31.625	-	31.625
6	31.643	-	31.646
	31.702	-	31.736
	31.625	33.790	31.625
10	31.643	34.058	31.643
	31.702	34.405	31.704
	31.625	32.695	31.625
20	31.643	32.710	31.643
	31.702	32.769	31.702
	31.625	31.740	31.625
40	31.643	31.763	31.643
	31.702	31.832	31.702

In the second test, three dimensionless frequencies parameters for two problems of beams resting on elastic foundation with the simply and the clamped support with various ratios of thickness to length were computed. The results of this analysis were provided in tables (2) and

(3). As shown in these tables, it is observed that the accuracy and convergence rate of the proposed element for the analysis of beams with various thicknesses is satisfactorily high.

Furthermore, the first four mode shapes corresponding to a thin beam (t/l =0.001) with simply-supported and clamped supports are shown in Fig. 3 and 4, respectively. These figures are plotted using 20 proposed elements. Moreover, in Fig. 5 the first dimensionless frequency was given as a function of the Winkler foundation parameter, for beams with different boundary conditions (i.e. clamped, simply, and simply-clamped beams).This figure reveals that beam with simply support is more sensitive to the variation in the foundation parameter $\bar{K}_w$.

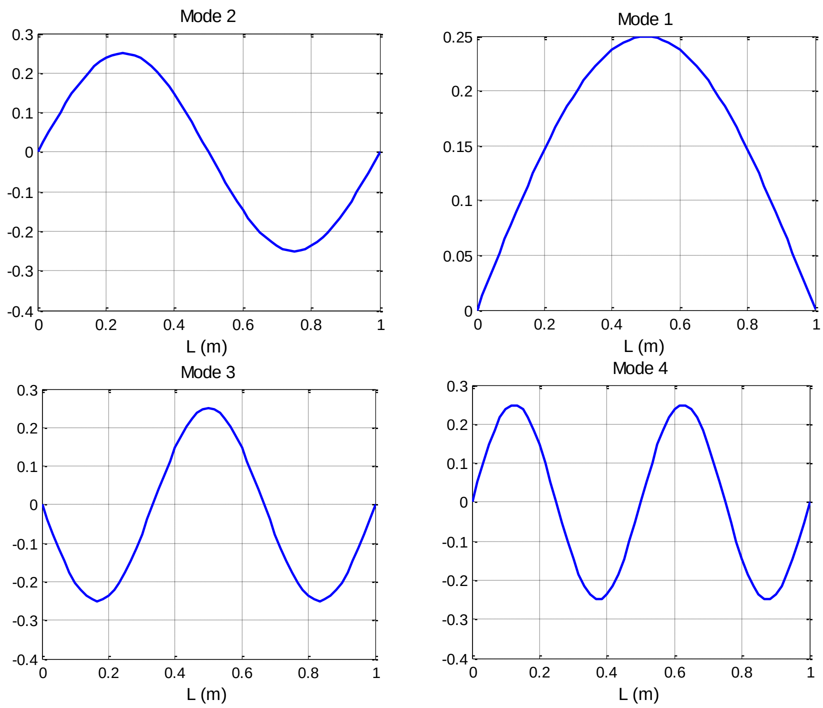
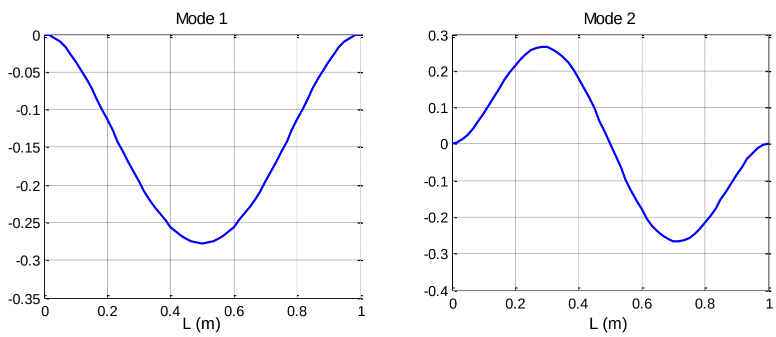


Figure 3. Mode shapes of simply-supported beam on Pasternak foundation ($\bar{K}_p$ = 2.5 and $\bar{K}_w$ = 10^6)



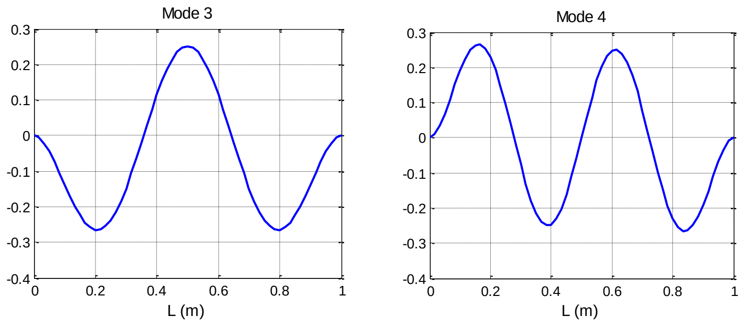


Figure 4. Mode shapes of clamped support beam on Pasternak foundation ($\bar{K}_p = 2.5$ and $\bar{K}_w = 10^6$)

Table 2: Convergence and accuracy of the first three dimensionless natural frequencies (Ω) for simply-supported beam

t/l	**Number of element**				**Chen et al. [13]**
	5	10	15	20	
1/120	4.1437	4.1436	4.1436	4.1436	4.1436
	6.7304	6.7263	6.7261	6.7261	-
	9.7344	9.7017	9.6997	9.6993	-
1/15	4.1363	4.1362	4.1362	4.1362	4.1347
	6.6595	6.6506	6.6495	6.6491	-
	9.5136	9.4484	9.4394	9.4365	-
1/5	4.0849	4.0841	4.0840	4.0839	4.0664
	6.2557	6.2250	6.2195	6.2176	-
	8.4869	8.3219	8.2912	8.2804	-

Table 3: Convergence and accuracy of the first three dimensionless natural frequencies (Ω) for clamped support beam

t/l	**Number of element**				**Chen et al. [13]**
	5	10	15	20	
1/120	5.1825	5.1815	5.1814	5.1814	5.1834
	8.1354	8.1214	8.1206	8.1205	8.1247
	11.2573	11.1878	11.1833	11.1825	11.1878
1/15	5.1260	5.1237	5.1234	5.1233	5.1254
	7.9180	7.8904	7.8868	7.8856	7.8928
	10.7764	10.6520	10.6340	10.6282	10.6388
1/5	4.8031	4.7950	4.7935	4.7930	4.7910
	6.9163	6.8501	6.8384	6.8343	6.8471
	8.9637	8.7357	8.6927	8.6778	7.4091

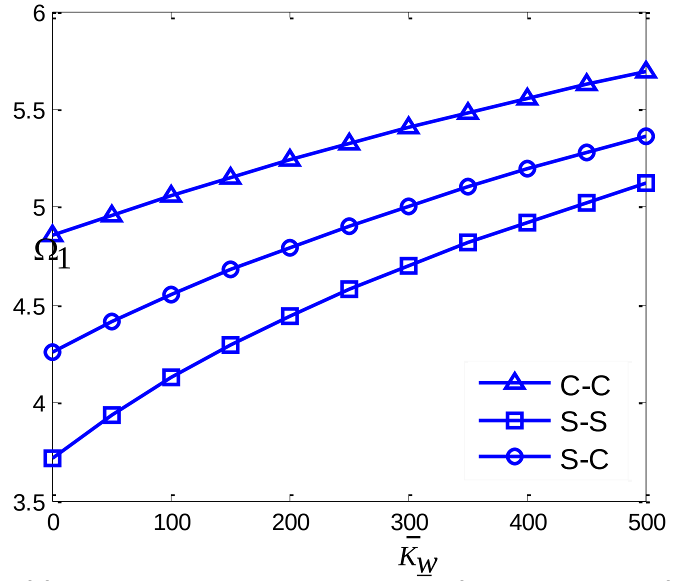


Figure 5. Curve of frequency parameter Ω_1 versus $\bar{K}_w$ for beams with different boundary conditions ($\bar{K}_p$ = 2.5 and *t/l* =0.001)

Finally, the free vibration analysis of thin beams (*t/l* =0.001) with different values of elastic foundation parameters was performed for simply and clamped support beams. The three dimensionless frequencies parameters Ω for these tests were shown in Tables (4) and (5) respectively and the results were compared with the findings of other researchers. To compare these results with that of reference [10], 15 proposed elements were utilized for the analysis. These tables reveal that the accuracy of the proposed element in the free vibration analysis of the beam on foundations with different characteristics is considerably high.

Table 4: The effect of foundation parameters on the three natural frequency parameters (Ω) for simply-supported beam

		Exact	**Chen et al. [13]**	**Rosa & Maurizi [10]**	**Proposed element**		
$\bar{K}_p$	$\bar{K}_w$	Ω_1	Ω_1	Ω_1	Ω_1	Ω_2	Ω_3
	0	3.1416	3.1414	3.1416	3.1416	6.2832	9.4248
0	100	3.7484	3.7482	3.7483	3.7484	6.3816	9.4545
	10000	10.0243	10.0240	10.0240	10.0243	10.3687	11.5652
	0	3.4767	3.4766	3.4767	3.4767	6.4709	9.553
0.5	100	3.9608	3.9607	3.9608	3.9608	6.5613	9.5816
	10000	10.0363	10.0361	10.0360	10.0363	10.4122	11.6354

	0	3.7360	3.7359	3.7360	3.7360	6.6437	9.6763
1	100	4.1437	4.1436	4.1437	4.1437	6.7273	9.7038
	10000	10.0484	10.0481	10.0480	10.0484	10.4550	11.7044
	0	4.2970	4.2969	4.2970	4.2970	7.0940	10.0204
2.5	100	4.5824	4.5823	4.5824	4.5824	7.1630	10.0452
	10000	10.0842	10.0839	10.0840	10.0842	10.5806	11.9042

Table 5: The effect of foundation parameters on the three natural frequency parameters (Ω) for clamped support beam

		Exact (Rosa & Maurizi [10])			**Chen et al. [13]**			**Proposed element**		
$\bar{K}_p$	$\bar{K}_w$	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
	0	4.73	7.854	10.996	4.7314	7.8533	10.9908	4.7300	7.8532	10.9956
0	100	4.95	7.904	11.014	4.9515	7.9044	11.0096	4.9504	7.9043	11.0144
	10000	10.123	10.839	12.526	10.1227	10.8384	12.5216	10.1229	10.8392	12.5260
	0	4.868	7.968	11.086	4.8683	7.9680	11.0815	4.8670	7.9678	11.0862
0.5	100	5.071	8.017	11.104	5.0718	8.0169	11.0998	5.0707	8.0168	11.1045
	10000	10.137	10.883	12.588	10.1373	10.8827	12.5832	10.1374	10.8835	12.5876
	0	4.994	8.078	11.174	4.9938	8.0777	11.1700	4.9926	8.0775	11.1747
1	100	5.182	8.124	11.192	5.1834	8.1247	11.1878	5.1824	8.1245	11.1926
	10000	10.152	10.927	12.648	10.1517	10.9264	12.6439	10.1518	10.9272	12.6483
	0	5.32	8.381	11.43	5.3195	8.3812	11.4233	5.3184	8.3811	11.4279
2.5	100	5.477	8.423	11.444	5.4783	8.4234	11.4400	5.4773	8.4232	11.4446
	10000	10.194	11.055	12.825	10.1942	11.0539	12.8209	10.1943	11.0546	12.8252

5. CONCLUSION

In this paper, an efficient finite element method for free vibration analysis of Timoshenko beams on elastic foundation was presented. The Pasternak assumption was applied with an aim to model the foundation. This model was found to provide relatively more accurate results. Also, a two-node Timoshenko element was introduced for the beam on the foundation. The stiffness matrices of the beam and foundation as well as the mass matrix of the beam element were also explicitly computed. Finally, in order to demonstrate the capability of the proposed element, numerical tests were performed for the free vibration analysis of beams with simply and clamped supports. These tests demonstrated the high accuracy and efficiency of the proposed element for free vibration analysis of beams on elastic foundation.

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